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Coulomb Systems Seen as Critical Systems: Finite-Size Effects in Two Dimensions

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It is known that the free energy at criticality of a finite two-dimensional system of characteristic size L has in general a term which behaves like $\log L$ as $L \to \infty$; the coefficient of this term is universal. There are solvable models of two-dimensional classical Coulomb systems which exhibit the same finite-size correction (except for its sign) although the particle correlations are short-ranged, i.e., noncritical. Actually, the electrical potential and electrical field correlations *are* critical at all temperatures (as long as the Coulomb system is a conductor), as a consequence of the perfect screening property of Coulomb systems. This is why Coulomb systems have to exhibit critical finite-size effects.

KEY WORDS: Two-dimensional critical systems; conformal invariance; Coulomb systems; finite-size effects; solvable models.

1. INTRODUCTION

Some time ago, it was shown by Cardy and Peschel[1,2] that the free energy, *at criticality*, of a finite two-dimensional system of characteristic size L has a term which behaves like $\log L$ as $L \to \infty$. More precisely, the free energy F (times the inverse temperature β) has a large-L expansion of the form

$$\beta F = AL^2 + BL - \frac{c\chi}{6} \log L + \cdots \tag{1.1}$$

The first two terms represent respectively the bulk free energy and the "surface" free energy; the corresponding coefficients A and B are nonuniversal.

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However, more interestingly, the coefficient of $\log L$ is universal, depending only on the conformal anomaly number c of the theory and on the Euler number χ of the manifold on which the system lives. This Euler number is completely defined by the topology of the manifold: $\chi = 2 - 2h - b$, where h is the number of handles and b the number of boundaries. For instance, $\chi = 2$ for a sphere, $\chi = 1$ for a disk, $\chi = 0$ for an annulus or a torus. The above results hold for a smooth metric and a smooth boundary.

An especially simple example is the Gaussian model, the partition function Z_G of which is defined in terms of a field $\phi(\mathbf{r})$ by the functional integral

$$Z_G = \int \mathscr{D}\phi \exp(-\beta \mathscr{H}_G) \tag{1.2}$$

with a Hamiltonian

$$\mathscr{H}_G = \frac{1}{4\pi} \int (\nabla\phi)^2 \, d^2\mathbf{r} \tag{1.3}$$

This model is critical at all temperatures, with a conformal anomaly number $c = 1$, and its free energy is indeed of the form (1.1).

While computing the free energy of finite two-dimensional Coulomb gases, in all cases when an exact solution has been obtained, we noted that this free energy was of the form

$$\beta F = AL^2 + BL + \frac{\chi}{6} \log L \tag{1.4}$$

strongly reminiscent of (1.1) for the Gaussian model case ($c = 1$), except for a change of sign. This critical-like behavior of the free energy of a Coulomb gas is at first sight unexpected, since the correlations between the particles are short-ranged, with a finite correlation length. However, a critical-like behavior of the free energy has already been found and explained by Forrester[3] for Coulomb gases with a periodic boundary condition; the essence of the explanation was that the Coulomb interaction ($\log r$ in two dimensions) is the inverse of the Laplacian operator Δ, and therefore a Coulomb gas is related to the Hamiltonian (1.3) rewritten as $-(4\pi)^{-1} \int \phi \, \Delta\phi \, d^2\mathbf{r}$.

One purpose of the present paper is to emphasize further the relevance of the Hamiltonian (1.3) for describing a Coulomb gas; indeed, if ϕ is interpreted as the electrical potential, (1.3) is just the well-known expression of the electrostatic energy in terms of the electrical field $\mathbf{E} = -\nabla\phi$. Usually, a Coulomb system is described in terms of the charge density $\rho(\mathbf{r})$. However, a description in terms of the electrical field $\mathbf{E}$ is the one which is appropriate for showing that a Coulomb system *is* from that alternative point of view a critical system. In Section 2 we discuss in more detail the description of a Coulomb system in terms of the electrical field $\mathbf{E}$ and how this leads to the term $(\chi/6)\log L$ in the free energy (1.4) of a finite system.

In Sections 3 and 4 we check the validity of (1.4) on two solvable models: the one-component plasma and the two-component plasma, at a special temperature. Another approach to the two-component plasma problem is discussed in the Appendix.

2. COULOMB SYSTEMS SEEN AS CRITICAL SYSTEMS

2.1. Coulomb Systems and Charge Fluctuations

The Coulomb systems that we consider here are made of charged particles which are assumed to move in a plane, for the time being. The interaction between two particles of charges q_i and q_j located at $\mathbf{r}_i$ and $\mathbf{r}_j$ is $-q_i q_j \log(|\mathbf{r}_i - \mathbf{r}_j|/a)$, where a is an arbitrary length; some short-range potential may be added. We shall also deal with a simplified model, the one-component plasma (OCP) or jellium, made of one species of particles of the same charge q embedded in a background of uniform charge density of the opposite sign. The systems under consideration always have zero total charge.

In the usual approach, an important quantity is the (microscopic) charge density

$$\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i)$$

(for the OCP model, the background charge density must be added). The simplest averages associated with $\rho(\mathbf{r})$ are $\langle \rho(\mathbf{r}) \rangle$ $(=0$ by neutrality) and the two-point function $S(|\mathbf{r} - \mathbf{r}'|) = \langle \rho(\mathbf{r})\, \rho(\mathbf{r}') \rangle$. A Coulomb system may have phase transitions. Here we consider only the case when the system is in a conducting phase. Then $S(r)$ is short-ranged, with a correlation length of the order of the average interparticle distance. The system is assumed to have good screening properties for the charges, and that results in the two Stillinger–Lovett sum rules[4]

$$\int S(r)\, d^2\mathbf{r} = 0, \qquad \beta \int S(r) r^2\, d^2\mathbf{r} = -2/\pi$$

In terms of the Fourier transform

$$S(k) = \int S(r) \exp(i\mathbf{k} \cdot \mathbf{r})\, d^2\mathbf{r}$$

the Stillinger–Lovett sum rules are equivalent to the statement that the small-k behavior of $S(k)$ is

$$S(k) \sim k^2/2\pi\beta \qquad (k \to 0)$$

i.e., that the two-point function of

$$\rho(\mathbf{k}) = \int \rho(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r})\, d^2\mathbf{r}$$

has the small-k behavior

$$\langle \rho(\mathbf{k})\, \rho(-\mathbf{k}') \rangle \sim (2\pi)^2\, \delta(\mathbf{k} - \mathbf{k}') k^2/2\pi\beta \qquad (k, k' \to 0) \tag{2.1}$$

Actually, the standard proof[3] of (2.1) can be extended, resulting into a stronger statement: For small k, *the random variables* $\rho(\mathbf{k})$ *are jointly Gaussian*, with the covariance matrix (2.1). This can be shown as follows. One starts with the assumption that an external charge density $\int \rho_{ext}(\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{r})\, d^2\mathbf{k}/(2\pi)^2$ is perfectly screened, i.e., that it induces in the system an opposite charge density, provided the external charge has only small-k Fourier components $\rho_{ext}(\mathbf{k})$. Since the external charge couples to the microscopic charge density $\rho(\mathbf{k})$ by an energy $\int \rho_{ext}(-\mathbf{k})(2\pi/k^2)\, \rho(\mathbf{k})\, d^2\mathbf{k}/(2\pi)^2$, the above assumption can be written

$$\frac{\langle \exp[-\beta \int \rho_{ext}(-\mathbf{k}')(2\pi/k'^2)\, \rho(\mathbf{k}')\, d^2\mathbf{k}'/(2\pi)^2]\, \rho(\mathbf{k}) \rangle}{\langle \exp[-\beta \int \rho_{ext}(-\mathbf{k}')(2\pi/k'^2)\, \rho(\mathbf{k}')\, d^2\mathbf{k}'/(2\pi)^2] \rangle} = -\rho_{ext}(\mathbf{k}) \tag{2.2}$$

where $\langle \cdots \rangle$ stands for an average with respect to the unperturbed Boltzmann factor. If $\rho_{ext}(\mathbf{k})$ is assumed to be infinitesimal, one obtains (2.1). The Gaussian behavior of $\rho(\mathbf{k})$ stems from the assumption that (2.2) holds also for a finite $\rho_{ext}(\mathbf{k})$. This can be seen by rewriting (2.2) in terms of a functional derivative as

$$\begin{aligned}\frac{\delta}{\delta\rho_{ext}(-\mathbf{k})} \log \left\langle \exp\left[-\beta \int \rho_{ext}(-\mathbf{k}') \left(\frac{2\pi}{k'^2}\right) \rho(\mathbf{k}') \frac{d^2\mathbf{k}'}{(2\pi)^2}\right] \right\rangle \\ = \frac{2\pi\beta}{k^2} \frac{1}{(2\pi)^2} \rho_{ext}(\mathbf{k})\end{aligned}$$

an equation which can be integrated into

$$\begin{aligned}\left\langle \exp\left[-\int \rho_{ext}(-\mathbf{k}) \frac{2\pi\beta}{k^2} \rho(\mathbf{k}) \frac{d^2\mathbf{k}}{(2\pi)^2}\right] \right\rangle \\ = \exp\left[\frac{1}{2} \int \rho_{ext}(-\mathbf{k}) \frac{2\pi\beta}{k^2} \rho_{ext}(\mathbf{k}) \frac{d^2\mathbf{k}}{(2\pi)^2}\right]\end{aligned} \tag{2.3}$$

The validity of (2.3) for arbitrary parameters $\rho_{\text{ext}}(-\mathbf{k})$ is the well-known characterization of a Gaussian behavior for the random variables $\rho(\mathbf{k})$, with the covariance (2.1).

Incidentally, the Gaussian behavior of the long-wavelength fluctuations holds as well for three-dimensional Coulomb systems. The present results are related to some previous ones.[5,6]

2.2. Electrical Potential and Electrical Field Fluctuations

The Fourier component $\phi(\mathbf{k})$ of the electrical potential is related to $\rho(\mathbf{k})$ by the Poisson equation $k^2\phi(\mathbf{k}) = 2\pi\rho(\mathbf{k})$. From (2.1) we obtain

$$\beta\langle\phi(\mathbf{k})\,\phi(-\mathbf{k}')\rangle \sim (2\pi)^2\,\delta(\mathbf{k}-\mathbf{k}')\,2\pi/k^2 \qquad (k, k' \to 0)$$

and therefore, for the electrical potential $\phi(\mathbf{r})$ in position space,

$$\beta\langle\phi(\mathbf{r})\,\phi(\mathbf{r}')\rangle \sim -\log|\mathbf{r}-\mathbf{r}'| + \text{const} \qquad (|\mathbf{r}-\mathbf{r}'| \to \infty) \tag{2.4}$$

From (2.4), one obtains for the electrical field $-\partial_\mu\phi(\mathbf{r})$

$$\beta\langle\partial_\mu\phi(\mathbf{r})\,\partial_\nu\phi(\mathbf{r}')\rangle \sim \left[\delta_{\mu\nu} - 2\,\frac{(\mathbf{r}-\mathbf{r}')^\mu\,(\mathbf{r}-\mathbf{r}')^\nu}{|\mathbf{r}-\mathbf{r}'|^2}\right]\frac{1}{|\mathbf{r}-\mathbf{r}'|^2} \qquad (|\mathbf{r}-\mathbf{r}'| \to \infty) \tag{2.5}$$

[The asymptotic behaviors (2.4) and (2.5) have been derived in a more careful way in ref. 7.] Furthermore, since the fluctuations are Gaussian (for small k, i.e., for large $|\mathbf{r}-\mathbf{r}'|$), one can use Wick's theorem for obtaining the n-point functions.

Thus, the asymptotic forms of these electrical potential and field n-point functions are the n-point functions of the Gaussian model (1.2). Now, the meaning of "asymptotic" is that the distances under consideration should be large compared to the characteristic microscopic length of the system, which is the average interparticle distance. But in a critical system such as the Gaussian model, one also has to introduce some short-distance cutoff. Therefore, in the Coulomb system, the average interparticle distance plays the role of the cutoff, and for larger distances the n-point functions are just identical in a Coulomb system and in the Gaussian model, and in that sense a Coulomb system *is* a critical system. Let us stress again that this is a consequence of perfect screening, as explained in Section 2.1. Incidentally, in that same spirit of disregarding the microscopic detail, (2.1) gives for the charge density 2-point function in position space

$$\beta\langle\rho(\mathbf{r})\,\rho(\mathbf{r}')\rangle = -\frac{1}{2\pi}\,\Delta\delta(\mathbf{r}-\mathbf{r}') \tag{2.6}$$

The Gaussian partition function (1.2) generates correlation functions which are those of the Coulomb system. However, this does *not* mean that one can compute the free energy of the Coulomb system by the direct use of (1.2), for a variety of reasons and in particular because (1.2) contains the functional integration $\int \mathscr{D}\phi \cdots$ rather than the integration upon the particle coordinates $\int d^2\mathbf{r}_1\, d^2\mathbf{r}_2\, d^2\mathbf{r}_3 \cdots$. Fortunately, we can circumvent that difficulty by working with the stress tensor, as follows.

2.3. Stress Tensor

The stress tensor, a generalization of the pressure, expresses the response of the system to an infinitesimal deformation. Following refs. 1 and 2, we consider a coordinate transformation $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(\mathbf{r})$, with α^μ infinitesimal. When the change of the Hamiltonian can be written, to first order in α^μ, as

$$\delta\mathscr{H} = -\frac{1}{2\pi}\int T_{\mu\nu}(\mathbf{r})\, \partial^\mu \alpha^\nu(\mathbf{r})\, d^2\mathbf{r} \tag{2.7}$$

this defines the stress tensor $T_{\mu\nu}(\mathbf{r})$.

The transformation $\mathbf{r} \to \mathbf{r}'$ can be interpreted either in an active sense (the system is deformed) or in a passive sense (the grid which defines the coordinates is deformed). We follow the choice made in refs. 1 and 2, which is the second one, i.e., $\delta\mathscr{H}$ is defined by

$$\mathscr{H}[\phi(\mathbf{r}')] + \delta\mathscr{H} = \mathscr{H}[\phi(\mathbf{r})] \tag{2.8}$$

We shall now show that the stress tensor is the same one for the Gaussian model and for a Coulomb system, except for a change of sign.

In the case of the Gaussian model Hamiltonian (1.3),

$$\mathscr{H}_G[\phi(\mathbf{r}')] = \frac{1}{4\pi}\int [\nabla'\phi(\mathbf{r}')]^2\, d^2\mathbf{r}', \qquad \mathscr{H}_G[\phi(\mathbf{r})] = \frac{1}{4\pi}\int [\nabla\phi(\mathbf{r})]^2\, d^2\mathbf{r}$$

and one readily finds that $\delta\mathscr{H}$ is of the form (2.7) with

$$T_{\mu\nu} = -\partial_\mu\phi\, \partial_\nu\phi + \tfrac{1}{2}\delta_{\mu\nu}(\nabla\phi)^2 \tag{2.9}$$

In the case of a Coulomb system, we can write the Hamiltonian in terms of the microscopic charge density $\rho(\mathbf{r})$ as

$$\mathscr{H}_C[\rho(\mathbf{r})] = -\frac{1}{2}\int \rho(\mathbf{r}_1) \log\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{a}\, \rho(\mathbf{r}_2)\, d^2\mathbf{r}_1\, d^2\mathbf{r}_2 \tag{2.10}$$

In doing so, we have included in $\mathscr{H}_C$ the self-energies of the particles; this is convenient for the purpose of the present paper. In terms of the transformed variables $\mathbf{r}'_i = \mathbf{r}_i + \boldsymbol{\alpha}(\mathbf{r}_i)$ $(i = 1, 2)$,

$$\mathscr{H}_C[\rho(\mathbf{r}')] = -\frac{1}{2}\int \rho(\mathbf{r}'_1) \log \frac{|\mathbf{r}'_1 - \mathbf{r}'_2|}{a} \rho(\mathbf{r}'_2)\, d^2\mathbf{r}'_1\, d^2\mathbf{r}'_2$$

Noting that by charge conservation $\rho(\mathbf{r}'_i)\, d^2\mathbf{r}'_i = \rho(\mathbf{r}_i)\, d^2\mathbf{r}_i$, expanding $\log(|\mathbf{r}_1 - \mathbf{r}_2|/a) = \log(|\mathbf{r}_1 - \mathbf{r}_2 + \boldsymbol{\alpha}(\mathbf{r}_1) - \boldsymbol{\alpha}(\mathbf{r}_2)|/a)$ to first order in $\boldsymbol{\alpha}$, introducing the electrical potential

$$\phi(\mathbf{r}_1) = -\int \log \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{a} \rho(\mathbf{r}_2)\, d^2\mathbf{r}_2$$

using the Poisson equation $\Delta\phi(\mathbf{r}) = -2\pi\rho(\mathbf{r})$ and the symmetry between $\mathbf{r}_1$ and $\mathbf{r}_2$, and defining $\delta\mathscr{H}_C$ in a way similar to (2.8), we obtain

$$\delta\mathscr{H}_C = \mathscr{H}_C[\rho(\mathbf{r})] - \mathscr{H}_C[\rho(\mathbf{r}')] = \frac{1}{2\pi}\int \Delta\phi(\mathbf{r})\, \partial_\mu\phi(\mathbf{r})\, \alpha^\mu(\mathbf{r})\, d^2\mathbf{r}$$

Using the identity

$$\Delta\phi\, \partial_\mu\phi = \partial^\nu[\partial_\mu\phi\, \partial_\nu\phi - \tfrac{1}{2}\delta_{\mu\nu}(\nabla\phi)^2]$$

we find after an integration by parts

$$\delta\mathscr{H}_C = -\frac{1}{2\pi}\int T_{\mu\nu}(\mathbf{r})\, \partial^\nu\alpha^\mu(\mathbf{r})\, d^2\mathbf{r}$$

with a stress tensor which now is

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi - \tfrac{1}{2}\delta_{\mu\nu}(\nabla\phi)^2 \tag{2.11}$$

i.e., has the opposite sign of the stress tensor (2.9) of the Gaussian model. Of course, (2.11) is just the Maxwell stress tensor.

The close relationship between the stress tensors is obviously related to the well-known transformation which allows one to reexpress the Coulomb energy (2.10) in terms of the electrical field $-\nabla\phi$, which gives the Gaussian Hamiltonian (1.3):

$$\mathscr{H}_C = \frac{1}{2}\int \rho(\mathbf{r})\, \phi(\mathbf{r})\, d^2\mathbf{r} = -\frac{1}{4\pi}\int [\Delta\phi(\mathbf{r})]\, \phi(\mathbf{r})\, d^2\mathbf{r}$$

$$= \frac{1}{4\pi}\int [\nabla\phi(\mathbf{r})]^2\, d^2\mathbf{r} = \mathscr{H}_G$$

The sign difference between the stress tensors can be understood if we now imagine the system as built on some elastic membrane which is deformed: it is not the same to have the charges or to have the potentials stuck on that membrane.

As defined here, the stress tensor (2.11) of the Coulomb gas is traceless, i.e., in the present approach one finds zero for the pressure p of an infinite Coulomb gas, while the known equation of state is[8]

$$\beta p = n\left(1 - \frac{\beta q^2}{4}\right)$$

where $\pm q$ is the charge of a particle and n is the number of particles per unit area. Indeed, quantities which depend on n, i.e., on the microscopic length $n^{-1/2}$, are necessarily lost in the present formalism, which correspondingly cannot account for the coefficients A and B of the free energy (1.4). On the contrary, the present formalism will account for the last term of (1.4).

2.4. Finite-Size Correction to the Free Energy

Since Coulomb systems have the same Hamiltonian, the same correlation functions, and the same stress tensor (except for its sign) as the Gaussian model, the calculations in refs. 1 and 2 are valid for the two-dimensional Coulomb gases, and they lead to (1.4), with $(\chi/6)\log L$ instead of $-(\chi/6)\log L$ accounted for by the different sign of the stress tensor. There is no need to reproduce here the details of these calculations of refs. 1 and 2. We shall only, for the sake of completeness, sketch the main points.

It is convenient to use the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. The correlation functions (2.4) and (2.5) become

$$\beta\langle\phi(\mathbf{r})\,\phi(\mathbf{r}')\rangle = -\frac{1}{2}\log(z - z') - \frac{1}{2}\log(\bar{z} - \bar{z}') + \text{const} \qquad (2.12)$$

$$\beta\langle\partial_z\phi(\mathbf{r})\,\partial_{z'}\phi(\mathbf{r}')\rangle = -\frac{1}{2}\frac{1}{(z - z')^2}$$

$$\beta\langle\partial_{\bar{z}}\phi(\mathbf{r})\,\partial_{\bar{z}'}\phi(\mathbf{r}')\rangle = -\frac{1}{2}\frac{1}{(\bar{z} - \bar{z}')^2}$$

$$\langle\partial_z\phi(\mathbf{r})\,\partial_{\bar{z}'}\phi(\mathbf{r}')\rangle = 0 \qquad (2.13)$$

and the nonvanishing elements of the traceless stress tensor are

$$T \equiv T_{zz} = (\partial_z\phi)^2, \qquad \bar{T} \equiv T_{\bar{z}\bar{z}} = (\partial_{\bar{z}}\phi)^2 \qquad (2.14)$$

An important part is played by the correlations of the stress tensor with itself. This is a 4-point function with coincident points, which is easily computed from the 2-point functions (2.13) by using Wick's theorem, with the regularization prescription that contractions between fields at the same point must be omitted. The nonvanishing elements are

$$\beta^2\langle T(\mathbf{r})\, T(\mathbf{r}')\rangle = \frac{1}{2(z-z')^4}, \qquad \beta^2\langle \bar{T}(\mathbf{r})\, \bar{T}(\mathbf{r}')\rangle = \frac{1}{2(\bar{z}-\bar{z}')^4} \tag{2.15}$$

These correlations characterize a conformally invariant theory with a conformal anomaly number $c=1$.

The above relations hold for an infinite flat Coulomb system. For dealing with a system living on a curved manifold, one treats the curvature as a perturbation which changes the metric, i.e., which behaves like a strain $\partial^\mu \alpha^\nu$ in (2.7). The linear response of the stress tensor to the corresponding perturbation Hamiltonian $\delta\mathscr{H}$ can be expressed in terms of the correlation functions (2.15). In this way, one obtains the average of the stress tensor of the curved system. That tensor now has a nonzero trace Θ and the average $\langle\Theta\rangle$ is related to the change $L(\partial F/\partial L)$ of the free energy F under a global dilatation (i.e. a change of the characteristic size L), which leads to the $\log L$ term in (1.4).

Similar considerations apply to the case when the Coulomb system has a (curved) boundary.[3] One starts with a semiinfinite system confined to the upper half-plane, and one introduces a curvature of the boundary as a perturbation. One then needs stress tensor correlations which generalize (2.15) to the case of the semiinfinite system.

For the Gaussian model, one finds that, in addition to (2.15), the correlations have the nonvanishing element

$$\beta^2\langle T(\mathbf{r})\, \bar{T}(\mathbf{r}')\rangle = \frac{1}{2(z-\bar{z}')^4} \tag{2.16}$$

in the upper half-plane. With the additional ingredient (2.16) one gets the general result (1.1), which includes the curvature effects of both the manifold and its boundary.

The analog of (2.16) for a Coulomb system can be found by using known features[(9)] of the charge–charge correlation function near a straight boundary (here the x axis). This function has an algebraic decay along the wall: for large $|x-x'|$ it has the asymptotic behavior

$$\beta\langle \rho(\mathbf{r})\, \rho(\mathbf{r}')\rangle \sim \frac{f(y, y')}{(x-x')^2}, \qquad |x-x'| \to \infty$$

[3] The boundary is assumed to be a plain hard wall which confines the charges. This does not generate any simple boundary conditions for the electrical potential.

where $f(y, y')$ is localized near the boundary $y = y' = 0$ and obeys the sum rule

$$\int dy \int dy' \, f(y, y') = -\frac{1}{2\pi^2}$$

If the microscopic detail is disregarded, one obtains a surface contribution to the charge fluctuations

$$\beta \langle \rho(\mathbf{r}) \, \rho(\mathbf{r}') \rangle = -\frac{1}{2\pi^2} \frac{\delta(y) \, \delta(y')}{(x - x')^2}$$

In other words, there is a surface charge density $\sigma(x)$ along the x axis, with correlations

$$\beta \langle \sigma(x) \, \sigma(x') \rangle = -\frac{1}{2\pi^2} \frac{1}{(x - x')^2} \tag{2.17}$$

The 2-point function of the electrical potential must be associated to a 2-point function of the charge, which now has both a bulk contribution (2.6) (in the upper half-plane only) and a surface contribution (2.17). One finds

$$\beta \langle \phi(\mathbf{r}) \, \phi(\mathbf{r}') \rangle = -\tfrac{1}{2} \log[(z - z')(\bar{z} - \bar{z}')] \quad \text{if} \quad y, y' > 0 \quad \text{or} \quad yy' < 0 \tag{2.18a}$$

$$\beta \langle \phi(\mathbf{r}) \, \phi(\mathbf{r}') \rangle = -\tfrac{1}{2} \log[(z - \bar{z}')(\bar{z} - z')] \quad \text{if} \quad y, y' < 0 \tag{2.18b}$$

It is easy to check that $\langle \rho(\mathbf{r}) \, \rho(\mathbf{r}') \rangle = (2\pi)^{-2} \langle \Delta\phi(\mathbf{r}) \, \Delta'\phi(\mathbf{r}') \rangle$ is indeed of the form (2.6) in the upper half-plane, while the discontinuities of $\langle \partial_y \phi(\mathbf{r}) \, \partial_{y'} \phi(\mathbf{r}') \rangle$ on the x axis do generate the surface charge correlations (2.17).

From (2.18b), one finds that the electrical field 2-point function in the *lower* half-plane is

$$\beta \langle \partial_z \phi(\mathbf{r}) \, \partial_{\bar{z}'} \phi(\mathbf{r}') \rangle = -\frac{1}{2} \frac{1}{(z - \bar{z}')^2}$$

Through Wick's theorem, one obtains (2.16) now in the *lower* half-plane. (We are now dealing with a Coulomb system, and although the charges are confined to the upper half-plane, the electrical potential and electrical field do not vanish in the lower half-plane.) Using (2.16), now in the lower half-plane, in the calculations of refs. 1 and 2, will generate the term $(\chi/6) \log L$ in (1.4).

Finally, it should be noted that the analogy between a Coulomb system and the Gaussian model will hold on a curved manifold only if the Coulomb interaction is defined in a way which preserves Poisson's equation $\Delta\phi(\mathbf{r}) = -2\pi\rho(\mathbf{r})$. For instance, for a Coulomb gas made of particles of charge q_i on the surface of a sphere, the potential must be chosen[10] as $\phi(\mathbf{r}) = -\sum_i q_i \log(|\mathbf{r}-\mathbf{r}_i|/a)$, with $|\mathbf{r}-\mathbf{r}_i|$ the length of the *chord* which joins surface points $\mathbf{r}$ and $\mathbf{r}_i$. It is then easy to check that

$$\Delta\phi(\mathbf{r}) = -2\pi \sum_i q_i \delta(\mathbf{r}-\mathbf{r}_i)$$

(where Δ is the two-dimensional Laplacian *on the sphere* and δ the two-dimensional "delta function," on the sphere), provided that the system is globally neutral, i.e., $\sum_i q_i = 0$, a condition which will always be assumed to hold.

The conclusion of the present section is that a finite two-dimensional Coulomb system, when in a conducting phase, has a free energy of the form (1.4). We shall now check this general formula in exactly solvable cases.

3. ONE-COMPONENT PLASMA

The two-dimensional, one-component plasma was defined in Section 2.1. It happens to be an exactly solvable model[11] in the canonical ensemble at that temperature such that $\beta q^2 = 2$, for a few geometries. Let us consider in turn these geometries.

3.1. The Disk

From the calculations in ref. 11, when $\beta q^2 = 2$ the free energy F for a system with N particles in a disk of radius R is given by

$$\beta F = \beta F_0 - \log Q - \log Q' \tag{3.1}$$

where βF_0 is the ideal gas part

$$\beta F_0 = -\log \frac{1}{N!}\left(\frac{\pi R^2}{\Lambda^2}\right)^N$$

(Λ is the de Broglie wavelength),

$$Q = \left[\exp\left(\frac{3}{4}N^2 - N\log\frac{R}{a}\right)\right] \frac{1!\,2!\,3!\cdots N!}{N^{N(N+1)/2}}$$

and

$$Q' = \prod_{n=0}^{N-1} \left[\frac{1}{n!} \int_0^N e^{-t} t^n \, dt \right]$$

In ref. 11, only the thermodynamic limit of F/N was computed. Here, we revisit that calculation, in order to obtain finite-size corrections. For a given value of the number density $n = N/\pi R^2$, we look for a large-R (or equivalently a large-N) expansion of βF.

Using Stirling's formula

$$\log N! = N \log N - N + \frac{1}{2} \log N + \frac{1}{2} \log(2\pi) + \frac{1}{12N} + \cdots \tag{3.2}$$

we find

$$\beta F_0 = N[\log(n\Lambda^2) - 1] + \tfrac{1}{2} \log N + O(1)$$

For computing $\log Q$, one can use the identity

$$\log(1!\, 2! \cdots N!) = (N+1) \log N! - \sum_{k=1}^{N} k \log k \tag{3.3}$$

and the expansion[12]

$$\sum_{k=1}^{N} k \log k = \frac{N^2}{2} \log N - \frac{N^2}{4} + \frac{1}{2} N \log N + \frac{1}{12} \log N + \frac{1}{12} - \zeta'(-1) + O\left(\frac{1}{N}\right) \tag{3.4}$$

(where ζ' is the derivative of Riemann's zeta function: $\zeta'(-1) = -0.1654...$) with the result

$$-\log Q = N[-\tfrac{1}{2} \log(\pi n a^2) + 1 - \tfrac{1}{2} \log(2\pi)] - \tfrac{5}{12} \log N + O(1)$$

As to $\log Q'$, it can be expanded by noting that $\int_0^N e^{-t} t^n \, dt$ is close to $n!$ except when n approaches N. The relevant contributions to $\log Q'$ are such that $N - n$ is of the order of $\sqrt{N}$, in which case one can use the asymptotic formula[13]

$$\frac{1}{n!} \int_0^N e^{-t} t^n \, dt = \frac{1}{2} \left[1 + \Phi\left(\frac{N-n}{(2N)^{1/2}} \right) \right] + O\left(\frac{1}{\sqrt{N}} \right)$$

where

$$\Phi(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} \, dx$$

is the error function. Replacing the sum on n by an integral on $y=(N-n)/(2N)^{1/2}$, we find

$$-\log Q' = -\sum_{n=0}^{N-1} \log\left(\frac{1}{n!}\int_0^N e^{-t} t^n \, dt\right)$$
$$= -(2N)^{1/2} \int_0^\infty dy \log \frac{1+\Phi(y)}{2} + O(1)$$

[the existence of a contribution $O(\sqrt{N})$ from $\log Q'$ was not properly mentioned in Eq. (2.19) of ref. 11].

Finally, the total free energy is given by

$$\beta F = \beta f \pi R^2 + \beta \gamma 2\pi R + \tfrac{1}{6} \log[(\pi n)^{1/2} R] + O(1) \qquad (3.5)$$

where the coefficients of πR^2 and $2\pi R$, respectively, involve the bulk free energy per unit area f and the surface tension γ, for which we recover the previously known expressions[11, 14]

$$\beta f = \left[\log(n\Lambda^2) - \frac{1}{2}\log(\pi n a^2) - \frac{1}{2}\log(2\pi)\right] n \qquad (3.6)$$

$$\beta\gamma = -(n/2\pi)^{1/2} \int_0^\infty dy \log \frac{1+\Phi(y)}{2} = 0.239 \left(\frac{n}{\pi}\right)^{1/2} \qquad (3.7)$$

The next term in (3.5) is $(1/6) \log R$, in agreement with the general formula (1.4), since for a disk $\chi = 1$.

3.2. The Annulus

The disk calculation can be generalized to the case of an annulus. The background and the N particles are now confined on an annulus with an inner radius R_1 and an outer radius R_2. The particle number density now is $n = N/\pi(R_2^2 - R_1^2)$ and it is convenient to define $N_1 = \pi n R_1^2$ and $N_2 = \pi n R_2^2$; thus $N_2 - N_1 = N$.

The total potential energy, including background–background, background–particle, and particle–particle interactions, is found to be

$$V = \frac{q^2}{2}\left[-\frac{N}{2}\log \pi n a^2 + \frac{1}{2} N_2^2 \log N_2 - \frac{1}{2} N_1^2 \log N_1 - \frac{3}{4}(N_2^2 - N_1^2) + \sum_{i=1}^{N} (\pi n r_i^2 - N_1 \log \pi n r_i^2) - \sum_{1 \leqslant i < j \leqslant N} \log \pi n r_{ij}^2\right]$$

where r_i is the distance of the ith particle to the center and r_{ij} the distance between particles i and j. Using the same method as in ref. 11, one obtains a free energy given by (3.1), where now

$$\beta F_0 = -\log \frac{1}{N!}\left[\frac{\pi(R_2^2 - R_1^2)}{\Lambda^2}\right]^N$$

$$Q = \left[\exp\left(\frac{3}{4}N_2^2 - \frac{3}{4}N_1^2 - \frac{1}{2}N_2^2 \log N_2 + \frac{1}{2}N_1^2 \log N_1 + \frac{N}{2}\log \pi n a^2\right)\right] \frac{N!}{N^N} N_1!\,(N_1+1)! \cdots (N_2-1)!$$

and

$$Q' = \prod_{n=N_1}^{N_2-1} \left(\frac{1}{n!}\int_{N_1}^{N_2} e^{-t} t^n \, dt\right)$$

Expansions of βF_0 and $-\log Q$ can be obtained by the same methods as in the case of the disk. For the calculation of $-\log Q'$, the relevant contributions are now from two domains of n: $N_2 - n = O(\sqrt{N_2})$ and $n - N_1 = O(\sqrt{N_1})$, which gives

$$-\log Q' = -[(2N_2)^{1/2} + (2N_1)^{1/2}] \int_0^\infty dy \log \frac{1 + \Phi(y)}{2} + O(1)$$

One finds a total free energy given by

$$\beta F = \beta f \pi (R_2^2 - R_1^2) + \beta \gamma 2\pi (R_1 + R_2) + \frac{1}{6}\log \frac{R_2}{R_1} + O(1) \tag{3.8}$$

where the bulk free energy per unit area f and the surface tension γ are again given by (3.6) and (3.7). Now, however, as R_2 and R_1 go to infinity with a fixed ratio R_2/R_1, the term $(1/6)\log(R_2/R_1)$ of (3.8) remains $O(1)$, as expected from (1.4) where $\chi = 0$ for an annulus.

3.3. The Sphere

As stated in Section 2.4, the interaction between two particles of a one-component plasma on the surface of a sphere must be chosen as $-q^2 \log(r/a)$, where r is the length of the chord which joins the two particles; the same prescription must be used for computing the background–background and particle–background interactions. For $\beta q^2 = 2$, an exact expression of the free energy has been previously found.[15] For N

particles on the surface of a sphere of radius R, the total free energy, including the ideal gas part, is given by

$$\beta F = -\log e^{N^2/2}\left(\frac{2\pi a R}{\Lambda^2}\right)^N \prod_{k=1}^{N} \frac{(k-1)!\,(N-k)!}{N!}$$

Again using (3.2)–(3.4), for a fixed value of the density $n = N/4\pi R^2$, we obtain the large-system expansion

$$\beta F = \beta f 4\pi R^2 + \tfrac{1}{3}\log[(4\pi n)^{1/2} R] + \tfrac{1}{12} - 2\zeta'(-1) + o(1)$$

with the same bulk free energy per unit area (3.6) as previously, no boundary term, and a $(1/3)\log R$ term in agreement with the general formula (1/4) where $\chi = 2$ for a sphere.

4. TWO-COMPONENT PLASMA

The two-dimensional, two-component plasma is a system of particles of charges q and $-q$; two particles located at $\mathbf{r}_i$ and $\mathbf{r}_j$ interact through a potential $\pm q^2 \log(|\mathbf{r}_i - \mathbf{r}_j|/a)$. This model is an exactly solvable one,[16–18] in the grand-canonical ensemble, at that temperature such that $\beta q^2 = 2$. Instead of the usual fugacity ζ, it is convenient to use a rescaled one, $m = 2\pi a \zeta$ (m is an inverse length). For a system confined in a finite two-dimensional plane domain D, the grand potential Ω is given by

$$\beta\Omega = -\mathrm{Tr}\log(1+K) = -\sum_i \log(1+\lambda_i) \tag{4.1}$$

where K is an integral operator with eigenvalues λ_i; the operator K, its eigenfunctions (ψ, χ), and its eigenvalues λ are defined by the two coupled integral equations

$$\frac{m}{2\pi}\int_D d^2\mathbf{r}' \frac{1}{z - z'}\chi(\mathbf{r}') = \lambda\psi(\mathbf{r}) \tag{4.2a}$$

$$\frac{m}{2\pi}\int_D d^2\mathbf{r}' \frac{1}{\bar{z} - \bar{z}'}\psi(\mathbf{r}') = \lambda\chi(\mathbf{r}) \tag{4.2b}$$

Actually, the trace in (4.1) diverges, and one must regularize it by introducing some cutoff. The physical interpretation is that the point-particle model with Coulomb interactions is unstable against the collapse of pairs of oppositely charged particles, unless the Coulomb interaction is regularized at short distance.

The integral equations (4.2) can be converted into differential equations with appropriate boundary conditions. Indeed, since

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z-z'} = \frac{\partial}{\partial z} \frac{1}{\bar{z}-\bar{z}'} = \pi\delta(\mathbf{r}-\mathbf{r}')$$

by applying the appropriate differential operators on (4.2) one finds the coupled equations

$$2\frac{\partial\chi(\mathbf{r})}{\partial z} = \frac{m}{\lambda}\psi(\mathbf{r}), \qquad 2\frac{\partial\psi(\mathbf{r})}{\partial \bar{z}} = \frac{m}{\lambda}\chi(\mathbf{r}) \tag{4.3}$$

(they can be rewritten in a more compact form in terms of the Dirac operator $\not{\partial}$; this is the well-known equivalence between the Coulomb gas at $\beta q^2 = 2$ and a free Fermi field). Equations (4.3) can be combined into the Laplacian eigenvalue problem

$$-\Delta\psi(\mathbf{r}) = k^2\psi(\mathbf{r}) \tag{4.4}$$

where $k^2 = -(m/\lambda)^2$. The boundary conditions can be obtained by a continuation of (4.2) for $\mathbf{r}$ outside the domain D: in each connected part of the exterior of D, ψ and χ are seen to be analytical functions of z and $\bar{z}$, respectively, vanishing at infinity in that connected part which is infinite; furthermore, ψ and χ must be continuous on each boundary. Thus the boundary conditions for (4.4) in D are $\psi|_{\partial D} = g$ and $\partial_{\bar{z}}\psi|_{\partial D} = \bar{h}$, where g and h are analytical in each connected part of the exterior of D and tend to zero at infinity if that connected part is infinite.

The operator $-\Delta$ with these boundary conditions can be shown to be self-adjoint; thus the eigenvalues λ are of the form $\pm i(m^2/k^2)^{1/2}$, and (4.1) can be rewritten as

$$\beta\Omega = -\operatorname{Tr}\log\left(1+\frac{m^2}{-\Delta}\right) = -\sum_i \log\left(1+\frac{m^2}{k_i^2}\right) \tag{4.5}$$

where the k_i^2 are the eigenvalues of $-\Delta$ with the previously described boundary conditions.

In the following, we shall consider a few geometries for which the grand potential can be studied in more detail, and we shall check that it has an expansion similar to (1.4):

$$\beta\Omega = AL^2 + BL + \frac{\chi}{6}\log L + \cdots \tag{4.6}$$

The proof that (4.6) should hold in general would be the same as the one sketched in Section 2.4. One would relate $L(\partial\Omega/\partial L)$ to the average $\langle\Theta\rangle$ of the trace of the stress tensor.

4.1. The Disk

Let the domain D be a disk of radius R. The eigenfunctions of (4.4) can be written in polar coordinates (r, φ) in terms of Bessel functions J_l as

$$\psi(\mathbf{r}) = J_l(kr)e^{il\varphi}$$

Then

$$\partial_{\bar{z}}\psi(\mathbf{r}) = -\frac{k}{2} J_{l+1}(kr)e^{i(l+1)\varphi}$$

If $l \geqslant 0$, the boundary conditions become $J_l(kR) = 0$ [then $g = 0$ and $\bar{h} = -(k/2)\, J_{l+1}(kR)(Re^{i\phi}/r)^{l+1}$]. If $l < 0$, the boundary conditions become $J_{l+1}(kR) = 0$ [then $g = J_l(kR)(re^{i\varphi}/R)^l$ and $\bar{h} = 0$]. Since $J_{l+1} = (-1)^{l+1} J_{-l-1}$, altogether (4.5) becomes

$$\beta\Omega = -2 \sum_{l=0}^{\infty} \sum_{n} \log\left(1 + \frac{m^2}{k_{l,n}^2}\right) \tag{4.7}$$

where the sum on n runs on all the positive roots of $J_l(kR) = 0$.

This sum on n can be evaluated exactly by noting that the entire function

$$f(z) = l!\,(2/Rz)^l J_l(Rz) \tag{4.8}$$

[which is such that $f(0) = 1$] has the infinite product representation

$$f(z) = \prod_n \left(1 - \frac{z^2}{k_{l,n}^2}\right)$$

Thus the sum on n is just $\log f(im)$, and

$$\beta\Omega = -2 \sum_{l=0}^{\infty} \log[l!\,(2/mR)^l I_l(mR)] \tag{4.9}$$

where I_l is a modified Bessel function.

As explained above, the sum (4.9) is divergent, and we shall regularize it by introducing an upper cutoff N on l. Although the sum cannot be done exactly, we can obtain a few terms of its asymptotic expansion for large R.

One way of doing this is to use the Debye expansion[19] of $I_l(\alpha)$ (we set $mR = \alpha$). This expansion is the one which is appropriate here, since it is valid for large α *uniformly* in l. It gives

$$\log I_l(\alpha) = -\frac{1}{2}\log(2\pi) - \frac{1}{4}\log(\alpha^2 + l^2) + \eta(l, \alpha) + \frac{3t - 5t^3}{24l} + O\left(\frac{1}{\alpha^2 + l^2}\right) \tag{4.10}$$

where

$$\eta(l, \alpha) = (\alpha^2 + l^2)^{1/2} - l \sinh^{-1}(l/\alpha), \qquad t = l/(\alpha^2 + l^2)^{1/2}$$

The sum (4.9), cut off as $\sum_{l=0}^{N}$, can now be evaluated: $\Sigma \log l!$ is obtained by using (3.3) and (3.4), and $\sum \log I_l$ is obtained by using (4.10) and the Euler–MacLaurin summation formula

$$\sum_{l=0}^{N} f(l) = \int_0^N f(x)\,dx + \tfrac{1}{2}[f(0) + f(N)] + \tfrac{1}{12}[f'(N) - f'(0)] + \cdots$$

The result is

$$\beta\Omega = -\beta p \pi R^2 + \beta\gamma 2\pi R + \frac{1}{6}\log(mR) - \frac{5}{12} + \frac{1}{3}\log 2 - 2\zeta'(-1) + O\left(\frac{1}{mR}\right) \tag{4.11}$$

where

$$\beta p = \frac{m^2}{2\pi}\left(1 + \log\frac{2N}{mR}\right) \tag{4.12}$$

and

$$\beta\gamma = m\left(\frac{1}{4} - \frac{1}{2\pi}\right) \tag{4.13}$$

The only term in (4.11) which diverges as $N \to \infty$ is the first one; the limit $N \to \infty$ has been taken in the rest of the expansion (4.11). If the Coulomb interaction is cut off at some short distance σ, the corresponding cutoff N on l should be of the order of R/σ, and we recover for the pressure p the known expression[17,18]

$$\beta p = \frac{m^2}{2\pi}\left(\log\frac{1}{m\sigma} + \text{const}\right) \tag{4.14}$$

The next term in (4.11) involves a surface tension γ which is also the known one.[18] Finally, (4.11) has a term $(1/6)\log R$, in agreement with the general formula (4.6), with $\chi = 1$ for a disk.

4.2. The Annulus

The disk calculation can be generalized to the case of an annulus with an inner radius R_1 and an outer radius R_2. The eigenfunctions of (4.4) now are of the form

$$\psi(r, \varphi) = [AJ_l(kr) + BN_l(kr)]\, e^{il\varphi}$$

where J_l and N_l are Bessel functions of the first and second kind, respectively. If $l \geqslant 0$, the boundary conditions now become $\psi(R_2, \varphi) = 0$ and $\partial_{\bar{z}}\psi(R_1, \varphi) = 0$, i.e.,

$$AJ_l(kR_2) + BN_l(kR_2) = 0$$

$$AJ_{l+1}(kR_1) + BN_{l+1}(kR_1) = 0$$

Thus, the eigenvalues $k_{l,n}$ are the positive roots of

$$J_l(kR_2)\, N_{l+1}(kR_1) - J_{l+1}(kR_1)\, N_l(kR_2) = 0 \tag{4.15}$$

Using a symmetry between l and $-l-1$ for dealing also with the case $l < 0$, one finds again a grand potential given by (4.7), where now the $k_{l,n}$ are the positive roots of (4.15).

The sum on n in (4.7) can be again brought to the form $\log f(im)$ by using the infinite product representation of the entire function $f(z)$, which is now

$$f(z) = -\frac{\pi}{2} \frac{R_1^{l+1}}{R_2^l}\, z[J_l(R_2 z)\, N_{l+1}(R_1 z) - J_{l+1}(R_1 z)\, N_l(R_2 z)]$$

and after simple manipulations on $f(im)$ one obtains

$$\beta\Omega = -2 \sum_{l=0}^{\infty} \log \left\{ m \frac{R_1^{l+1}}{R_2^l} [I_l(mR_2)\, K_l'(mR_1) + I_l'(mR_1)\, K_l(mR_2)] \right\} \tag{4.16}$$

where I_l and K_l are modified Bessel functions.

Again we cut off the sum on l at some upper value N and we look for an asymptotic expansion of $\beta\Omega$ when $R_1, R_2 \to \infty$ with a fixed ratio R_2/R_1, using the Debye expansions[19] for I_l, I_l', K_l, K_l'. The term $I_l'K_l$ in (4.16) can be discarded because it is found to give a contribution exponentially smaller than the one from the term I_lK_l'. Using again the Euler–MacLaurin summation formula, one finally finds

$$\beta\Omega = -\frac{m^2}{2}\left[\left(1 + \log\frac{2N}{mR_2}\right) R_2^2 - \left(1 + \log\frac{2N}{mR_1}\right) R_1^2\right]$$
$$- m\left(1 - \frac{\pi}{2}\right)(R_1 + R_2) + \frac{1}{6}\log\frac{R_2}{R_1} + O\left(\frac{1}{mR_2}\right) \tag{4.17}$$

Again, the only divergent term in (4.17) is the first one. In terms of the short-distance cutoff σ of the Coulomb potential, the cutoff N must be chosen of the order of R/σ, where R is some characteristic size of the annulus. Extensitivity can be obtained by choosing

$$R = R_2 x^{-x^2/(1-x^2)}$$

with $x = R_1/R_2$. Then

$$\beta\Omega = -\beta p\pi(R_2^2 - R_1^2) + \beta\gamma 2\pi(R_1 + R_2) + \frac{1}{6}\log\frac{R_2}{R_1} + O\left(\frac{1}{mR_2}\right) \tag{4.18}$$

with a pressure p and a surface tension γ again given by (4.14) and (4.13), respectively. As R_1 and R_2 go to infinity with a fixed ratio x, the term $(1/6)\log(R_2/R_1)$ of (4.18) remains $O(1)$, in agreement with the general formula (4.6), where $\chi = 0$ for an annulus.[4]

4.3. The Sphere

The case of the sphere was considered in ref. 20. On a sphere of radius R, the grand potential is

$$\beta\Omega = -\frac{1}{2}\operatorname{Tr}\log\left(1 - \frac{m^2}{\not{D}^2}\right)$$

where $\not{D}$ is the Dirac operator on the sphere. Its eigenvalues are $\pm in/R$, where n is any positive integer, with multiplicity $2n$. Thus

$$\beta\Omega = -2\sum_{n=1}^{\infty} n\log\left(1 + \frac{m^2R^2}{n^2}\right)$$

Introducing an upper cutoff for n, one finds the asymptotic expansion

$$\beta\Omega = -\beta p 4\pi R^2 + \frac{1}{3}\log mR - 4\zeta'(-1) - \frac{1}{60m^2R^2} + \cdots$$

in agreement with the general formula (4.6), where $\chi = 2$ for a sphere.

5. CONCLUSION

The correlations of the electrical potential and electrical field in a classical Coulomb system (when in a conducting phase) are identical to the

[4] A more detailed account of the calculations in Sections 4.1 and 4.2 is available from C. Pisani on request.

corresponding ones for the Gaussian model of field theory (which is critical at all temperatures). This behavior of the correlation functions of Coulomb systems is equivalent to their perfect screening property.

From that point of view, Coulomb systems are critical systems. The universal finite-size correction to the free energy of two-dimensional critical systems does apply in general to Coulomb systems. This can be explicitly checked on solvable models.

APPENDIX. MORE ON THE TWO-COMPONENT PLASMA AT $\beta q^2 = 2$

An alternative approach to the finite-size, two-component plasma at $\beta q^2 = 2$ is obtained by studying the total number of particles N rather than the grand potential Ω. For a system confined to a plane domain D, one obtains from (4.5)

$$N = -m \frac{d}{dm}(\beta\Omega) = \mathrm{Tr}\,\frac{2m^2}{m^2 - \Delta} = 2m^2 \int_D d^2\mathbf{r}\; G(\mathbf{r}, \mathbf{r}) \tag{A.1}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green function of $m^2 - \Delta$, i.e., the solution of

$$(m^2 - \Delta)\, G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{A.2}$$

with the peculiar boundary conditions described at the beginning of Section 4.

In the special case of a domain D with circular symmetry (disk, annulus), one can obtain the solution of (A.2) in the form of an expansion in circular harmonics $e^{il(\varphi - \varphi')}$, compute the integral on $\mathbf{r}$ in (A.1), and integrate $N(m)/m$ with respect to m. This is another method for deriving (4.9) or (4.16). We leave it as an exercise for the reader.

One can look for a relation between the asymptotic expansion (4.6) and a general small-t expansion first studied by Kac[21] for the spectral function $\Phi_d(t)$ of the Laplacian Δ_d with *Dirichlet* boundary conditions in an arbitrary finite connected smooth domain D:

$$\Phi_d(t) = \mathrm{Tr}\, e^{-t(-\Delta_d)} = \frac{A}{4\pi t} - \frac{L}{8(\pi t)^{1/2}} + \frac{1}{6}(1 - r) + \cdots \tag{A.3}$$

where A is the area of D, L is the length of its perimeter, and r is the number of holes in D. Indeed, (A.1) can be written as

$$N = \mathrm{Tr}\,\frac{2m^2}{m^2 - \Delta} = 2m^2 \int_0^\infty dt\; e^{-m^2 t}\Phi(t) \tag{A.4}$$

where

$$\Phi(t) = \mathrm{Tr}\, e^{-t(-\Delta)} \tag{A.5}$$

However, the Laplacian which appears in (A.5) is the one with the boundary conditions of Section 4, and we have not been able to make a general connection between the present spectral function $\Phi(t)$ defined with these boundary conditions and the Dirichlet spectral function (A.3).

The connection can be made, however, in the special case of a disk, when the expression for N which corresponds to (4.7) and (4.9) is

$$\begin{aligned} N &= \mathrm{Tr}\, \frac{2m^2}{m^2 - \Delta} = 2 \sum_{l=0}^{\infty} \sum_n \frac{2m^2}{m^2 + k_{l,n}^2} \\ &= \sum_{l=-\infty}^{\infty} \sum_n \frac{2m^2}{m^2 + k_{l,n}^2} + \sum_n \frac{2m^2}{m^2 + k_{0,n}^2} \\ &= \mathrm{Tr}\, \frac{2m^2}{m^2 - \Delta_d} + \frac{mRI_0'(mR)}{I_0(mR)} \end{aligned}$$

where we have set aside an $l = 0$ term (there is only one such term for Dirichlet boundary conditions, while this term has a twofold degeneracy for the present boundary conditions). From (A.4),

$$N = 2m^2 \int_0^{\infty} dt\, e^{-m^2 t} \Phi_d(t) + \frac{mRI_0'(mR)}{I_0(mR)}$$

Using (A.3) for $\Phi_d(t)$ and the large-R expansion

$$\frac{mRI_0'(mR)}{I_0(mR)} = mR - \frac{1}{2} + \cdots$$

one gets

$$N = \frac{m^2 \pi R^2}{2\pi} \int^{\infty} \frac{dt}{t} e^{-m^2 t} - \frac{m 2\pi R}{4} + \frac{1}{3} + \cdots + mR - \frac{1}{2} + \cdots$$

When regularized by the introduction of a small-t cutoff, the integral in the first term of N is of the order of $\log(1/m^2\sigma^2)$, and by an integration of $N(mR)/mR$ with respect to mR one recovers the expansion (4.11) of the grand potential up to the $\log(mR)$ term. The extra $l = 0$ term contributes to both the surface tension (4.13) and to the universal $\log(mR)$ term. Note that the constant term in (4.11) cannot be obtained by the present method, because this constant is lost when one looks for an asymptotic expansion of the derivative $d\Omega/dm$ rather than of Ω itself.

Since one expects the asymptotic expansion (4.6) to be valid for a smooth domain D of arbitrary shape, with shape-independent pressure, surface tension, and $(\chi/6)\log L$ terms, one also expects that the spectral function (A.5) with the boundary conditions of Section 4 has the corresponding shape-independent small-t expansion

$$\Phi(t)=\frac{A}{4\pi t}-\left(\frac{1}{8}-\frac{1}{4\pi}\right)\frac{L}{(\pi t)^{1/2}}-\frac{1}{12}(1-r)+\cdots$$

Such an expansion might be hidden somewhere in the mathematical literature.

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